Advances and Applications in Mathematical Sciences
Volume 21, Issue 3, January 2022, Pages 1259-1269

THE FORCING MONOPHONIC GLOBAL DOMINATION NUMBER OF A GRAPH

V. SELVI[1] and V. SUJIN FLOWER[2]

[1]Research Scholar, Reg.No.20123042092008
[2]Assistant Professor
Department of Mathematics
Holy Cross College (Autonomous)
Nagercoil, 629 004, India
Affiliated to Manonmaniam Sundaranar University
Abishekapatti, Tirunelveli - 627 012
Tamil Nadu, India
E-mail: selvi.maths1983@gmail.com
sujinflower@gmail.com

Abstract

Let G be a connected graph and let S be a minimum monophonic global dominating set of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique minimum monophonic global dominating set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing monophonic global domination number of S, denoted by $f_{\bar{\gamma}m}(S)$, is the cardinality of a minimum forcing subset of S. The forcing monophonic global domination number of G, denoted by $f_{\bar{\gamma}m}(G)$, is $f_{\bar{\gamma}m}(G) = \min\{f_{\bar{\gamma}m}(S)\}$, where the minimum is taken over all minimum monophonic global dominating sets S in G. Some of its general properties are studied. It is shown that for every pair a, b of integers with $0 \leq a \leq b$, there exists a connected graph G such that $f_{\bar{\gamma}m}(G) = a$ and $\bar{\gamma}_m(G) = b$, where $\bar{\gamma}_m(G)$ is the global domination number of a graph.

1. Introduction

Let $G = (V, E)$ be a graph with a vertex set $V(G)$ and edge set $E(G)$ (or simply V and E, respectively, Furthermore, we say that a graph G has order

2020 Mathematics Subject Classification: 05CS38, 05CS69.
Keywords: Forcing monophonic global dominating number; Monophonic global domination number, Monophonic number; Global domination number, Domination number.
Received July 8, 2021; Accepted October 11, 2021

$n = |V(G)|$ and size $m = |E(G)|$. For basic graph theoretic terminology, we refer to [2]. A vertex v is adjacent to another vertex u if and only if there exists an edge $e = uv \in E(G)$. If $uv \in E(G)$, we say that u is a neighbor of v and denote by $N_G(v)$, the set of neighbors of v. The degree of a vertex $v \in V$ is $\deg_G(v) = |N_G(v)|$. A vertex v is said to be a universal vertex if $\deg_G(v) = n - 1$. A vertex v is called an extreme vertex if the sub graph induced by v is complete. The length of a path is the number of its edges. Let u and v be vertices of a connected graph G. A shortest $u - v$ path is also called a $u - v$ geodesic. The (shortest path) distance is defined as the length of a $u - v$ geodesic in G and is denoted by $d_G(u, v)$ or $d(u, v)$ for short if the graph is clear from the context. A chord of a path P is an edge which connects two non-adjacent vertices of P. An $u - v$ path is called a monophonic path if it is a chordless path. For two vertices u and v, the closed interval $J[u, v]$ consists of all the vertices lying in a $u - v$ monophonic path including the vertices u and v. If u and v are adjacent, then $J[u, v] = \{u, v\}$. For a set M of vertices, let $J[M] = \cup_{u, v \in M} J[u, v]$. Then certainly $M \subseteq J[M]$. A set $M \subseteq V(G)$ is called a monophonic set of G if $J[M] = V$. The monophonic number $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is called a m-set of G. The monophonic number of a graph was studied in [1, 5-13].

A subset $D \subseteq V(G)$ is called a dominating set if every vertex in $V \setminus D$ is adjacent to at least one vertex of D. The domination number, $\gamma(G)$, of a graph G denotes the minimum cardinality of such dominating sets of G. A minimum dominating set of a graph G is hence often called as a γ-set of G. The domination concept was studied in [3, 4]. A subset $D \subseteq V$ is called a global dominating set in G if D is a dominating set in G and $\overline{G}$. The global domination number $\bar{\gamma}(G)$ is the minimum cardinality of a minimum global dominating set in G. The concept of global domination in graph was introduced in [14]. A set $M \subseteq V$ is said to be a monophonic global dominating set of G if M is both a monophonic set and a global dominating set of G. The minimum cardinality of a monophonic global dominating set of G is the monophonic global domination number of G and is denoted by $\bar{\gamma}_m(G)$. A monophonic global dominating set of cardinality $\bar{\gamma}_m(G)$ is called a $\bar{\gamma}_m$-set of G. The concept of monophonic global domination in graph was introduced in [15].

A social network theory is concerned about the study of relationships between the members of a group. A social network clique is a dominating clique which is a group of representatives of the network who can communicate themselves directly. A status in a network is a subset S of members of the group such that any two of them have the same relationship outside S in the network. A group of people equivalent if any two of them have same relationship between people in the social network. Kelleher and Cozzens exhibited that these sets can be determined using the properties of dominating sets. Concepts related to minimal path also appear in the social sciences. When members of a social group are represented by vertices, and particular relationships between members by edges of a graph, then the monophonic number can be considered as the smallest number p such that every member of a group is contained in a minimal chain of relationships between chosen p members.

The following theorem is used in sequel.

Theorem 1.1 [15]**.** *Each extreme vertex of a connected graph G belongs to every monophonic global dominating set of G.*

2. The Forcing Monophonic Global Domination Number of a Graph

Definition 2.1. Let G be a connected graph and let M be a minimum monophonic global dominating set of G. A subset $T \subseteq M$ is called a forcing subset for M if M is the unique minimum monophonic global dominating set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing monophonic global domination number of M, denoted by $f_{\bar{\gamma}m}(M)$, is the cardinality of a minimum forcing subset of M. The forcing monophonic global domination number of G, denoted by $f_{\bar{\gamma}m}(G)$, is $f_{\bar{\gamma}m}(G) = \min\{f_{\bar{\gamma}m}(M)\}$, where the minimum is taken over all minimum monophonic global dominating sets M in G.

Example 2.2. For the graph G given in Figure 2.1, $M_1 = \{v_1, v_3, v_5\}$, $M_2 = \{v_1, v_4, v_6\}$, $M_3 = \{v_2, v_4, v_5\}$, $M_4 = \{v_2, v_4, v_6\}$, $M_5 = \{v_2, v_4, v_7\}$ and $M_6 = \{v_2, v_5, v_7\}$ are the only six minimum monophonic global dominating sets of G such that $f_{\bar{\gamma}m}(M_1) = f_{\bar{\gamma}m}(M_2) = f_{\bar{\gamma}m}(M_3) = f_{\bar{\gamma}m}(M_4) = f_{\bar{\gamma}m}(M_5) = f_{\bar{\gamma}m}(M_6) = 2$ so that $f_{\bar{\gamma}m}(G) = 2$.

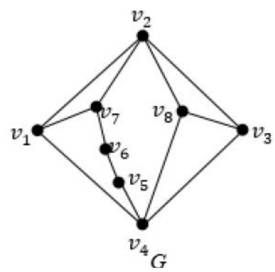

Figure 2.1

Definition 2.3. A vertex v is said to be a monophonic global dominating vertex of G if v belongs to every minimum monophonic global dominating set of G. If G has a unique minimum monophonic global dominating set M, then every vertex of M is a monophonic global dominating set of G.

Example 2.4. For the graph G given in Figure 2.2, $M_1 = \{v_2, v_5, v_7\}$ and $M_2 = \{v_2, v_4, v_7\}$ are the only two $\bar{\gamma}_m$-sets of G such that $X = \{v_2, v_7\}$ is the set of all monophonic global dominating vertices of G.

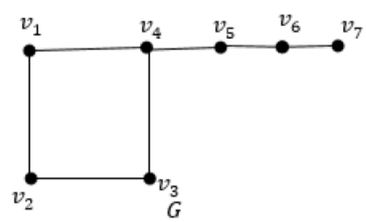

Figure 2.2

Note 2.5. Each extreme vertex and each universal vertex of G are monophonic global dominating vertices of G. In fact there are monophonic global dominating vertices which are neither extreme vertices of G nor universal vertices of G. For the graph G given in Figure 2.2, v_2 is a monophonic global dominating vertex of G which is neither an extreme vertex nor a universal vertex of G.

The next theorem follows immediately from the definition of the monophonic global domination number and the forcing monophonic global domination number of a connected graph G.

Observation 2.6. For every connected graph G, $0 \leq f_{\bar{\gamma}m}(G) \leq \bar{\gamma}_m(G)$.

Remark 2.7. The bounds in the Observation 2.6 are sharp. For the complete graph $G = K_n (n \geq 2)$, $M = V(G)$ is the unique $\bar{\gamma}_m$-set of G so that $f_{\bar{\gamma}m}(G) = 0$. For the cycle $G = C_5$ with $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $M_1 = \{v_1, v_2, v_3\}$, $M_2 = \{v_2, v_3, v_4\}$, $M_3 = \{v_3, v_4, v_5\}$, $M_4 = \{v_4, v_5, v_1\}$, $M_5 = \{v_5, v_1, v_2\}$, $M_6 = \{v_1, v_3, v_4\}$, $M_7 = \{v_2, v_4, v_5\}$, $M_8 = \{v_3, v_5, v_1\}$, $M_9 = \{v_4, v_1, v_2\}$ and $M_{10} = \{v_5, v_2, v_3\}$ are the only ten $\bar{\gamma}_m$-sets of G such that $f_{\bar{\gamma}m}(M_i) = 3$ and $\bar{\gamma}_m(G) = 3$ for $i = 1$ to 10 so that $f_{\bar{\gamma}m}(G) = 3$. Also the bounds are strict. For the graph in Figure 2.1, $\bar{\gamma}_m(G) = 3$, $f_{\bar{\gamma}m}(G) = 2$. Thus $0 < f_{\bar{\gamma}m}(G) < \bar{\gamma}_m(G)$.

Theorem: 2.8. *Let G be a connected graph. Then*

(a) $f_{\bar{\gamma}m}(G) = 0$ *if and only if G has a unique $\bar{\gamma}_m$-set.*

(b) $f_{\bar{\gamma}m}(G) = 1$ *if and only if G has at least two $\bar{\gamma}_m$-sets, one of which is a unique $\bar{\gamma}_m$-set containing one of its elements, and*

(c) $f_{\bar{\gamma}m}(G) = \bar{\gamma}_m(G)$ *if and only if no $\bar{\gamma}_m$-set of G is the unique $\bar{\gamma}_m$-set containing any of its proper subsets.*

Proof. (a) Let $f_{\bar{\gamma}m}(G) = 0$. Then, by definition, $f_{\bar{\gamma}m}(M) = 0$ for some $\bar{\gamma}_m$-set M of G so that the empty set ϕ is the minimum forcing subset for M. Since the empty set ϕ is a subset of every set, it follows that M is the unique $\bar{\gamma}_m$-set of G. The converse is clear.

(b) Let $f_{\bar{\gamma}m}(G) = 1$. Then by Theorem 2.8(a), G has at two least two $\bar{\gamma}_m$-sets. Also, since $f_{\bar{\gamma}m}(G) = 1$, there is a singleton subset T of a $\bar{\gamma}_m$-set M of G

such that T is not a subset of any other $\bar{\gamma}_m$-set of G. Thus M is the unique $\bar{\gamma}_m$-set containing one of its elements. The converse is clear.

(c) Let $f_{\bar{\gamma}m}(G) = \bar{\gamma}_m(G)$. Then $f_{\bar{\gamma}m}(M) = \bar{\gamma}_m(G)$ for every $\bar{\gamma}_m$-set M in G. Also, we know that $\bar{\gamma}_m(G) \geq 2$ and hence $f_{\bar{\gamma}m}(G) \geq 2$. Then by Theorem 2.7(a), G has at least two $\bar{\gamma}_m$-sets and so the empty set ϕ is not a forcing subset for any $\bar{\gamma}_m$-set of G. Since $f_{\bar{\gamma}m}(M) = \bar{\gamma}_m(G)$, no proper subset of M is a forcing subset of M. Thus no $\bar{\gamma}_m$-set of G is the unique $\bar{\gamma}_m$-set containing any of its proper subsets. Conversely, the data implies that G contains more than one $\bar{\gamma}_m$-set and no subset of any $\bar{\gamma}_m$-set M other than M is a forcing subset for M. Hence it follows that $f_{\bar{\gamma}m}(G) = \bar{\gamma}_m(G)$. ∎

Theorem 2.9. *Let G be a connected graph and let $\Im$ be the set of relative complements of the minimum forcing subsets in their respective $\bar{\gamma}_m$-sets in G. Then $\bigcap_{F \in \Im} F$ is the set of monophonic global dominating vertices of G.*

Proof. Let X be the set of all monophonic global dominating vertices of G. We are to show that $X = \bigcap_{F \in \Im} F$. Let $v \in X$. Then v is a monophonic global dominating of G that belongs to every $\bar{\gamma}_m$-set M of G. Let $T \subseteq M$ be any minimum forcing subset for any $\bar{\gamma}_m$-set M of G. We claim that $v \notin T$. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that M is the unique $\bar{\gamma}_m$-set containing T' so that T' is a forcing subset for M with $|T'| < |T|$, which is a contradiction to T is a minimum forcing subset for M. Thus $v \notin T$ and so $v \in F$, where F is the relative complement of T in M. Hence $v \in \bigcap_{F \in \Im} F$ so that $X \subseteq \bigcap_{F \in \Im} F$.

Conversely, let $v \in \bigcap_{F \in \Im} F$. Then v belongs to the relative complement of T in M for every T and every M such that $T \subseteq M$, where T is a minimum forcing subset for M. Since F is the relative complement of T in M, we have $F \subseteq M$ and thus $v \in M$ for every M, which implies that v is a monophonic global dominating vertex of G. Thus $v \in X$ and so $\bigcap_{F \in \Im} F \subseteq X$. Hence $X \subseteq \bigcap_{F \in \Im} F$. ∎

Theorem 2.10. *Let G be a connected graph and X be the set of all monophonic global dominating vertices of G. Then $f_{\bar{\gamma}m}(G) \leq \bar{\gamma}_m(G) - |X|$.*

Proof. Let M be any $\bar{\gamma}_m$-set of G. Then $\bar{\gamma}_m(G) = |M|$, $X \subseteq M$ and M is the unique $\bar{\gamma}_m$-set containing $M - X$. Thus $f_{\bar{\gamma}m}(G) \leq |M - X| = |M| - |X|$ $= |M| - |X| = \bar{\gamma}_m(G) - |X|$. ∎

Remark 2.11. The bounds in the Theorem 2.10 is sharp. For the G given in Figure 2.2, $f_{\bar{\gamma}m}(M_1) = f_{\bar{\gamma}m}(M_2) = 1$, $f_{\bar{\gamma}m}(G) = 1$ $\bar{\gamma}_m(G) = 3$ and $|X| = 2$. Thus $f_{\bar{\gamma}m}(G) = \bar{\gamma}_m(G) - |X|$.

In the following we determine the forcing monophonic global domination number of some standard graphs.

Observation 2.12 (i)**.** If G is either the complete graph $K_n (n \geq 2)$ or the star $K_{1,\, n-1} (n \geq 3)$, $f_{\bar{\gamma}m}(G) = 0$

(ii) For a double star G, $f_{\bar{\gamma}m}(G) = 0$.

Theorem 2.13. *For the cycle* $G = C_n (n \geq 4)$.

$$f_{\bar{\gamma}m}(G) = \begin{cases} 3 & if\ n = 4 \\ 1 & if\ n \equiv 0 \pmod 3 \\ 2 & otherwise \end{cases}$$

Proof. Let C_n be $v_1, v_2, \ldots, v_n, v_1$.

Case 1. Let $n = 4$. Then $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_2, v_3, v_4\}$ $S_3 = \{v_3, v_4, v_1\}$ and $S_4 = \{v_4, v_1, v_2\}$ are the only four minimum monophonic global dominating sets of G such that $f_{\bar{\gamma}m}(S_1) = f_{\bar{\gamma}m}(S_2)$ $= f_{\bar{\gamma}m}(S_3) = f_{\bar{\gamma}m}(S_4) = 3$ so that $f_{\bar{\gamma}m}(G) = 3$.

Case 2. Let $n = 5$. Then $S_1 = \{v_1, v_3, v_4\}$, $S_2 = \{v_1, v_2, v_4\}$ $S_3 = \{v_1, v_3, v_5\}$, $S_4 = \{v_2, v_4, v_5\}$ and $S_5 = \{v_2, v_3, v_5\}$ are the only five minimum monophonic global dominating sets of G such that $f_{\bar{\gamma}m}(S_i) = 2$ for $1 \leq i \leq 5$ so that $f_{\bar{\gamma}m}(G) = 2$.

Case 3. Let $n \equiv 0 \pmod 3$. Let $n = 3k$, $k \geq 2$. Then $S = \{v_1, v_4, v_7, \ldots, v_{3k-1}\}$ is the unique $\bar{\gamma}_m$-set of G containing $\{v_1\}$, so that $f_{\bar{\gamma}m}(G) = 1$.

Case 4. Let $n \equiv 1 \pmod 3$. Let $n = 3k + 1$, $k \geq 3$. Let S be any $\bar{\gamma}_m$-set of G. Then it is easily verified that any singleton subset of S is a subset of another $\bar{\gamma}_m$-set of G and so $f_{\bar{\gamma}m}(G) \geq 2$. Now, $S_1 = \{v_1, v_4, v_7, \ldots, v_{3k+1}\}$ is the unique $\bar{\gamma}_m$-set of G containing $\{v_1, v_{3k+1}\}$ so that $f_{\bar{\gamma}m}(G) = 2$.

Case 5. Let $n + 1 \equiv 0 \pmod 3$. Let $n = 3k - 1$, $k \geq 2$. Let S be any $\bar{\gamma}_m$-set of G. Then it is easily verified that any singleton subset of S is a subset of another $\bar{\gamma}_m$-set of G and so $f_{\bar{\gamma}m}(G) \geq 2$. Now, $S_1 = \{v_1, v_4, v_7, \ldots, v_{3k-2}\}$ is the unique $\bar{\gamma}_m$-set of G containing $\{v_1, v_{3k-2}\}$ so that $f_{\bar{\gamma}m}(G) = 2$. ■

Theorem 2.14. *For the path* $G = P_n (n \geq 4)$,

$$f_{\bar{\gamma}m}(G) = \begin{cases} 0 \text{ if } n - 1 \equiv 0 \ (mod\ 3) \text{ and } n = 4 \\ 1 \text{ if } n \equiv 0 \ (mod\ 3) \text{ and } n = 5 \\ 2 \text{ if } n + 1 \equiv 0 \ (mod\ 3) \text{ and } n \geq 8 \end{cases}$$

Proof. Let P_n be $v_1, v_2, v_3, \ldots, v_n$.

Case 1. Let $n = 4$. Then $S_1 = \{v_1, v_4\}$ is the unique $\bar{\gamma}_m$-set of G such that $f_{\bar{\gamma}m}(G) = 0$.

Case 2. Let $n - 1 \equiv 0 \pmod 3$. Let $n = 3k + 1$, $k \geq 2$. Then $S = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k+1}\}$ is the unique $\bar{\gamma}_m$-set of G so that $f_{\bar{\gamma}m}(G) = 0$.

Case 3. Let $n \equiv 0 \pmod 3$. Let $n = 3k$, $k \geq 2$. Then $S_1 = \{v_1, v_3, v_6\}$ and $S_2 = \{v_1, v_4, v_6\}$ are the only two $\bar{\gamma}_m$-set of G such that $f_{\bar{\gamma}m}(S_1) = f_{\bar{\gamma}m}(S_2) = 1$ so that $f_{\bar{\gamma}m}(G) = 1$. Next assume that $k \geq 3$, then $S = \{v_1, v_3, v_6, v_9, \ldots, v_{3k}\}$ is the unique $\bar{\gamma}_m$-set of G containing $\{v_3\}$ so that $f_{\bar{\gamma}m}(G) = 1$.

Case 4. Let $n + 1 \equiv 0 \pmod 3$. Let $n = 3k - 1$, First assume that $k = 2$. Then $S_1 = \{v_1, v_2, v_5\}$, $S_2 = \{v_1, v_3, v_5\}$ and $S_3 = \{v_1, v_4, v_5\}$ are the only three $\bar{\gamma}_m$-set of G such that $f_{\bar{\gamma}m}(S_1) = f_{\bar{\gamma}m}(S_2) = f_{\bar{\gamma}m}(S_3) = 1$ so that $f_{\bar{\gamma}m}(G) = 1$. Next assume that $k \geq 3$, then $S = \{v_1, v_4, v_7, \ldots, v_{3k-5}, v_{3k-2}, v_{3k-1}\}$, is the unique $\bar{\gamma}_m$ -set of G containing $\{v_{3k-5}, v_{3k-2}\}$ so that $f_{\bar{\gamma}m}(G) = 2.$ ■

Theorem 2.15. *For wheel* $G = K_1 + C_{n-1}(n \geq 5)$, $f_{\bar{\gamma}m}(G) = \begin{cases} 1 & \text{for } n = 5 \\ 2 & \text{for } n \geq 5 \end{cases}.$

Proof. Let $V(K_1) = x$ and $V(C_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$. For $n = 5$, $S_1 = \{x, v_1, v_3\}$ and $S_2 = \{x, v_2, v_4\}$ are the only two $\bar{\gamma}_m$ -sets of G so that $f_{\bar{\gamma}m}(G) = 1.$ So, let $n \geq 5.$ Then x is the monophonic global dominating vertex of G. Let $S_{ij} = \{x, u_i, v_j\}$ $(1 \leq i \neq j \leq n-1)$. Since $n \geq 5$, $\bar{\gamma}_m$ -set is not unique so that $f_{\bar{\gamma}m}(G) = 1.$ Now no singleton subsets of S_{ij} is a forcing subset of S_{ij} and $f_{\bar{\gamma}m}(G) \geq 2.$ Since S_{ij} is the unique $\bar{\gamma}_m$ -set of G containing $[u_i, v_j]$, it follows that $f_{\bar{\gamma}m}(G) = 2.$ ■

Theorem 2.16. *For the fan graph* $G = K_1 + P_{n-1}(n \geq 5)$, $f_{\bar{\gamma}m}(G) = 0.$

Proof. Let $V(K_1) = \{x\}$ and $V(P_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$. Then $Z = \{x, v_1, v_{n-1}\}$ is the set of monophonic global dominating vertices of G. By Theorem 1.1, Z is a subset of every monophonic global dominating set of G and so $\bar{\gamma}_m(G) \geq 3.$ Since $V(G) \setminus Z$, is dominated by an element of Z and $V(\overline{G}) \setminus Z$, is dominated by an element of Z, Z is a $\bar{\gamma}_m$ -set of G. Since Z is the unique $\bar{\gamma}_m$ -set of G, we have $f_{\bar{\gamma}m}(G) = 0.$ ■

In view of Observation 2.6, we have the following realization result.

Theorem 2.17. *For every pair of positive integers a and b with* $0 \leq a \leq b$ *and* $b \geq a + 1$, *there exists a connected graph G such that* $f_{\bar{\gamma}m}(G) = a$ *and* $\bar{\gamma}_m(G) = b$.

Proof. Let $P_i : u_i, v_i, w_i, y_i (1 \leq i \leq a)$ be a copy path of order 4. Let H be a graph obtained from $P_i (1 \leq i \leq a)$ by adding a new vertex x joining x with each $u_i (1 \leq i \leq a)$ and each $y_i (1 \leq i \leq a)$. Let G be a graph obtained from H a by adding new vertices $z_1, z_2, \ldots, z_{b-a-1}$ and joining x with each $z_i (1 \leq i \leq b-a-1)$. The graph G is shown in Figure 2.3.

First we show that $\bar{\gamma}_m(G) = b$. Let $Z = \{x, z_1, z_2, \ldots, z_{b-a-1}\}$ be the set of all monophonic global dominating vertices of G and $H_i = \{v_i, w_i\}$ $(1 \leq i \leq a)$. It is easily seen that every minimum monophonic dominating set of G contains at least one vertex from each $H_i (1 \leq i \leq a)$ and so $\bar{\gamma}_m(G) \geq 1 + b - a - 1 + a = b$. Now $S = Z \cup \{v_1, v_2, \ldots, v_a\}$ is a monophonic dominating set of G so that $\bar{\gamma}_m(G) = b$.

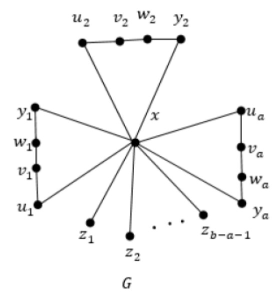

Figure 2.3

Next we show that $f_{\bar{\gamma}m}(G) = a$. By Observation 2.6, $f_{\bar{\gamma}m}(G) \leq \bar{\gamma}_m(G) - |Z| = a$. Now since $\bar{\gamma}_m(G) = b$ and every minimum monophonic global dominating set of G contains Z, it is easily seen that every $\bar{\gamma}_m$-set of G is of the form $S = Z \cup \{s_1, s_2, \ldots, s_a\}$, where $s_i \in H_i (1 \leq j \leq a)$. Let T be any proper subset of S with $|T| < a$. Then there exists a vertex $s_j (1 \leq j \leq a)$ such that $s_j \notin T$. Let t_j be a vertex of H_j distinct from s_j. Then $S_1 = \{S - \{s_j\} \cup \{t_j\}\}$ is a $\bar{\gamma}_m$-set of G properly containing T. Therefore T is not a forcing subset of S. This is true for all $\bar{\gamma}_m$-sets of G. Hence it follows that $f_{\bar{\gamma}m}(G) = a$. ■

References

[1] P. Arul Paul Sudhahar and C. Bency, The forcing restrained monophonic number of a graph, International Journal of Applied Engineering Research 13(16) (2018), 12449-12452.

[2] F. Buckley and F. Harary, Distance in Graph, Addition-Wesly-wood City, CA, (1990).

[3] G. Chartrand, H. Gavlas, R. C. Vandell and F. Harary, The forcing domination number of a graph, Journal of Combinatorial Mathematics and Combinatorial Computing 25

(1997), 161-174.

[4] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, (1998).

[5] J. John and P. A. P. Sudhahar, The forcing edge monophonic number of a graph, Scientia Series A: Mathematical Sciences 23 (2012), 87-98.

[6] J. John and S. Panchali, The forcing monophonic number of a graph, International Journal of Mathematical Archives (3) (2012), 935-938.

[7] J. John and V. Mary Gleeta, The forcing monophonic hull number of a graph, International Journal of Mathematics Trends and Technology 3(2) (2012), 43-46.

[8] J. John and P. Arul Paul Sudhahar, On the edge monophonic number of a graph, Filomat 26(6) (2012), 1081-1089.

[9] J. John and P. Arul Paul Sudhahar and A Vijayan, The connected monophonic number of a graph, International Journal of Combinatorial Graph Theory and Applications 5 (2012).

[10] J. John and P. Arul Paul Sudhahar, The forcing edge monophonic number of a graph, Scientia Series A: Mathematical Sciences 23 (2012), 87-98.

[11] J. John and V. Mary Gleeta, On the forcing hull and forcing monophonic hull numbers of graphs, Mathematical Combinatorics 3 (2012), 30-39.

[12] J. John, P. Arul Paul Sudhahar and D. Stalin, On the (M, D) number of a graph, Proyecciones Journal of Mathematics 38(2) (2019), 255-266.

[13] J. John, The forcing monophonic and the forcing geodetic numbers of a graph, Indonesian Journal of Combinatorics 4(2) (2020), 114-125.

[14] E. Sampath Kumar, The global domination number of a graph, J. Math. Phys. Sci. 23 (1989), 377-385.

[15] V. Selvi and V. Sujin Flower, The monophonic global domination number of a graph, J. Math. Comput. Sci. 11(5) (2021), 6007-6017.